Effects of Magnetic and Gravitational Torques on Spinning Satellite Attitude

MICHIO HARA*
Central Research Laboratory, Hitachi, Ltd., Kokubunji, Tokyo, Japan

An analysis of the rotational motion of a uniaxial satellite due to magnetic and gravitational torques is presented. The rotational motion of a spinning rigid body responding to these torques, averaged over arbitrary portions of an eccentric orbit, can be described by an autonomous system of nonlinear differential equations. From this the phase trajectory described by the spin vector is obtained to show that the motion is periodic in the phase space fixed to the orbit plane. Poincaré's method has been employed to analyze the perturbation effects of the gravitational torque on the periodic motion due to the dominant magnetic torque. It is found that the perturbed change in period is proportional to the projection of the spin vector on the characteristic vector about which the spin vector rotates. The approximate solution for the perturbed motion is also obtained as the coefficient of the power-series expansion in the small parameter representing the effect of the gravitational torque.

1. Introduction

SPIN stabilization for attitude control of Earth satellites aims to keep the spin axis inertially fixed, by utilizing the principle of conservation of angular momentum. In practice, however, long-term drift of the spin axis is inevitable, due to various unintended perturbing torques which exist in the space environment. But it is possible to make a purposeful use of these disturbances as torques to perform attitude maneuvers. One of the well-established methods which take advantage of the environment is a magnetic torquing method, which utilizes interaction of the Earth's magnetic field with a satellite's magnetic moment. This method, because of its simplicity and reliability, has been widely used with success on a number of low to medium altitude satellites.

Beletsky[1] was first to study the problem of the rotational motion of a uniaxial satellite in a circular orbit influenced by perturbing torques such as those produced by aerodynamic and gravitational factors. He introduced into the equations of motion for the angular momentum vector a term corresponding to the perturbation produced by the nodal regression of the orbit. However, the theoretical analysis of the rotational motion was, for the most part, carried out relative to the nonregressing orbit. The effects of the nodal regression were studied by employing a numerical approach.

Later Beletsky,[2,3] Colombo,[4] and Holland[5] developed analytical methods for predicting the motion of the angular momentum vector of uniaxial and triaxial rigid bodies relative to the regressing orbit. In so doing, they achieved good agreement with the observed motions of the Sputnik III, the Explorer, and the Pegasus satellites which are under the dominant effect of the gravitational torque.

In all of the aforementioned references, the external torques acting on the satellites are averaged over an entire orbit, because their interest is chiefly in the prediction of attitude variations of the satellites without special means for attitude control.

In Ref. 6 and also here, we have treated external torques as mean torques averaged over arbitrary portions of the satellite orbit. This treatment is essential in the establishing of command laws for an attitude control which utilizes the external torques acting on a satellite over specific portions of the orbit. Such a treatment may represent an attempt to give a more generalized form of the theoretical results on the gyroscopic behavior of a spinning satellite. The problem treated in Ref. 6 and here can be reduced to that of the attitude prediction considered in the aforementioned works, if the interval of integration is spread over an entire orbit.

As an extension to the works mentioned previously, the present paper aims at a more detailed analysis of effects of external torques on the attitude dynamics of a spin-stabilized satellite. The case considered here is that of a uniaxial satellite responding to the mean magnetic and gravitational torques averaged over arbitrary portions of an eccentric orbit. In deriving the equations of motion, it is assumed that the Earth's magnetic field is represented by an uncanted dipole and that the satellite angular momentum vector coincides with its spin axis. After deriving them, general phase trajectories[6] of the system and criteria deciding the shape of the trajectories are presented, as prerequisites to the discussions mentioned below.

Then primary attention has been centred upon the analysis of the perturbation effects of the gravitational torque on the rotational motion due to the dominant magnetic torque, by applying Poincaré's method concerning approximation of periodic solutions for nonlinear systems. Some theoretical results defining the interaction between the both torques in terms of the physical and orbital parameters of the system will be given. They are intended to serve as an aid in gaining a better understanding of the effects of the external torques on spinning satellite attitude.

2. Phase Trajectory

The equations for the rotational motion of a rigid body may be written as

$$d\mathbf{L}/dt = d^*\mathbf{L}/dt + \mathbf{\Omega} \times \mathbf{L} = \mathbf{T} \tag{1}$$

where $\mathbf{L}$ is the angular momentum, $\mathbf{T}$ is the external torque, $\mathbf{\Omega}$ is the angular velocity of a reference coordinate system, and d^*/dt is the time derivative relative to this coordinate system.

In deriving the equations of motion for a spinning uniaxial satellite, the assumption is made that the satellite angular momentum coincides with the spin vector at all times. Because the external torques acting on the satellite may be very small, this assumption is deemed reasonable in the presence of an internal damping mechanism. Thus

$$\mathbf{L} = I_z \omega_s \mathbf{x} \tag{2}$$

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* Research Engineer, 9th Department.

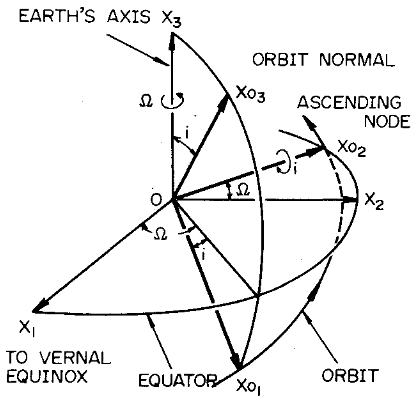

Fig. 1 Orbital coordinate system.

where I_z is the moment of inertia about the spin axis, ω_s is the satellite spin rate, and $\mathbf{x}$ is the unit vector along the spin axis.

The external torques to be considered here are magnetic and gravitational torques, which may be expressed as

$$\mathbf{T}_M = \mathbf{M}_s \times \mathbf{H} = M_s \mathbf{x} \times \mathbf{H} = (M_s m/r^3)\mathbf{x} \times \{\mathbf{X}_m - 3(\mathbf{r}\cdot\mathbf{X}_m)\mathbf{r}\} \tag{3}$$

and

$$\mathbf{T}_G = (3g_0 R_0{}^2/r^3)(I_z - I_x)(\mathbf{r}\cdot\mathbf{x})(\mathbf{r}\times\mathbf{x}) \tag{4}$$

where $\mathbf{T}_M$ is the magnetic torque, $\mathbf{M}_s$ is the magnetic moment along the spin axis, M_s is its magnitude, $\mathbf{H}$ is the Earth's magnetic field, m is the Earth's magnetic dipole, $\mathbf{X}_m$ is the unit vector along the axis of this dipole, r is the distance from the Earth's center to the satellite, $\mathbf{r}$ is the unit vector along this direction, $\mathbf{T}_G$ is the gravitational torque, g_o is the acceleration of gravity on the Earth's surface, R_o is the mean radius of the Earth, and I_x is the moment of inertia transverse to the spin axis.

These external torques are averaged over some portion of the orbit to obtain the mean external torques. Using the following relationships which hold for a Keplerian orbit $r = a(1-e^2)/(1+e\cos v)$, $dv/dt = h/r^2$, and $h = 2a^2(1-e^2)^{1/2}/T$ (where a is the semimajor axis, e is the eccentricity, v is the true anomaly of the satellite, h is the specific angular momentum, and T is the orbital period), we have the final results for the mean external torques:

$$(\mathbf{T}_{M\parallel})_{\text{mean}} = \frac{1}{t(v_f)-t(v_i)}\int_{t(v_i)}^{t(v_f)} \mathbf{T}_M\,dt = \frac{1}{t(v_f)-t(v_i)}\int_{v_i}^{v_f} \mathbf{T}_M \frac{dt}{dv}dv = M_s mk\cos\zeta\,\{(3I_1 - I_3)\sin i\mathbf{X}_{01} - 3I_2 \sin i\mathbf{X}_{02} + I_3\cos i\mathbf{X}_{03}\} \tag{5}$$

and

$$(\mathbf{T}_G)_{\text{mean}} = \frac{1}{t(v_f)-t(v_i)}\int_{t(v_i)}^{t(v_f)} \mathbf{T}_G\,dt = 3g_o R_o{}^2(I_z - I_x)k[\{I_1(\mathbf{X}_{01}\cdot\mathbf{x}) - I_2(\mathbf{X}_{02}\cdot\mathbf{x})\}\mathbf{X}_{01} + \{-I_2(\mathbf{X}_{01}\cdot\mathbf{x}) + (I_3 - I_1)(\mathbf{X}_{02}\cdot\mathbf{x})\}\cdot\mathbf{X}_{02}] \times \mathbf{x} \tag{6}$$

in which

$$k = T/2\pi a^3(1-e^2)^{3/2}\{t(v_f)-t(v_i)\} \tag{7}$$

and

$$\begin{cases} I_1 = \displaystyle\int_{v_i}^{v_f} \sin^2(v+P)(1+e\cos v)\,dv \\ I_2 = \displaystyle\int_{v_i}^{v_f} \sin(v+P)\cos(v+P)\,dv \\ I_3 = \displaystyle\int_{v_i}^{v_f} (1+e\cos v)\,dv \end{cases} \tag{8}$$

where $(\mathbf{T}_{M\parallel})_{\text{mean}}$ is the mean magnetic torque due to the Earth's dipole component along the Earth's axis, $t(v)$ is the time when the satellite passes through v, ζ is the angle between the Earth's axis and the axis of the Earth's dipole, i is the inclination, P is the argument of perigee, and the subscripts i and f indicate the initial and final conditions, respectively. The vectors $\mathbf{X}_{01}$, $\mathbf{X}_{02}$, and $\mathbf{X}_{03}$ are the unit vectors along the coordinate axes of the orbital coordinate system (X_{01}, X_{02}, X_{03}) as shown in Fig. 1, where the coordinate system (X_1, X_2, X_3) represents the inertial system having the unit vectors $\mathbf{X}_1$, $\mathbf{X}_2$, and $\mathbf{X}_3$, respectively.

The angular velocity $\mathbf{\Omega}$ on the left-hand side of Eq. (1) may ordinarily be taken as that of the nonspinning coordinate system which has a coordinate axis fixed on the spin axis, but does not rotate with the satellite. However, inspection of the form of expression for the averaged external torques [Eqs. (5) and (6)] may suggest it convenient to set $\mathbf{\Omega}$ equal to the angular velocity of the orbital coordinate system (X_{01}, X_{02}, X_{03}), in order to facilitate the analysis of the rotational motion, as is indicated later. Thus we set

$$\mathbf{\Omega} = -\Delta\Omega\mathbf{X}_3 = \Delta\Omega\sin i\mathbf{X}_{01} - \Delta\Omega\cos i\mathbf{X}_{02} \tag{9}$$

where $\Delta\Omega$ is the regression of the nodes.

By letting the components of spin vector along the X_{01}, X_{02}, and X_{03} axes be x, y, and z respectively, that is,

$$\mathbf{x} = x\mathbf{X}_{01} + y\mathbf{X}_{02} + z\mathbf{X}_{03} \tag{10}$$

and substituting Eqs. (2, 4, 5, and 9) into Eq. (1), the equations of motion follow as

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{g}(\mathbf{x}) \tag{11}$$

where

$$A = \begin{bmatrix} 0, & -d/2, & f \\ d/2, & 0, & -g \\ -f, & g, & 0 \end{bmatrix} \tag{12}$$

$$\mathbf{g}(\mathbf{x}) = \begin{bmatrix} (hx+by)z \\ -(ax+hy)z \\ -(hx+by)x + (ax+hy)y \end{bmatrix} \tag{13}$$

$$\begin{cases} a = 3GI \\ b = 3G(I_3 - I_1) \\ 2h = -6GI_2 \\ 2g = -2\{M(3I_1 - I_3) + \Delta\Omega\}\sin i \\ 2f = 6MI_2\sin i \\ d = -2(MI_3 - \Delta\Omega)\cos i \end{cases} \tag{14}$$

and

$$\begin{cases} G = g_o R_o{}^2(I_z - I_x)k/I_z\omega_s \\ M = M_s mk\cos\zeta/I_z\omega_s \end{cases} \tag{15}$$

where dot is d^*/dt, and G and M represent the constants related to the gravitational and magnetic torques, respectively.

The equations of motion derived above constitute a third-order autonomous system of nonlinear differential equations. The analysis of the rotational motion begins by obtaining the phase trajectory of the system. Upon integration of the equation

$$(ax+hy+g)\dot{x} + (hx+by+f)\dot{y} + (d/2)\dot{z} = 0 \tag{16}$$

which can be derived from the autonomous system of Eq. (11), we obtain

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c + dz = 0 \tag{17}$$

where c is an integration constant. The preceding equation represents a quadratic surface, and

$$ab - h^2 = 9G^2[I_1(I_3 - I_1) - I_2{}^2] > 0 \quad [\because \text{ from Eq. (8)},\ I_1 > I_2,\ (I_3 - I_1) > I_2] \tag{18}$$

which indicates that the quadratic surface is an elliptic paraboloid.

Another integration of the equation $x\dot{x} + y\dot{y} + z\dot{z} = 0$, which can also be obtained from Eq. (11), yields

$$x^2 + y^2 + z^2 = 1 \tag{19}$$

Hence, the phase trajectory of the autonomous system, the locus which the tip of the spin vector describes in the phase space fixed with the orbit plane, as shown in Fig. 2, is formed by the intersection of the elliptic paraboloid of Eq. (17) and the sphere of Eq. (19) (Ref. 6). This may indicate that the rotational motion of a spinning satellite is periodic.

Consider here the case where the external torques are averaged over an entire orbit. By setting $v_i = 0$ and $v_f = 2\pi$ in Eq. (8), one obtains $I_1 = \pi$, $I_2 = 0$, and $I_3 = 2\pi$. Consequently, there follows from Eq. (14) $a = b = 3\pi G$ and $h = f = 0$, which may imply that Eq. (17) represents a rotary elliptic paraboloid the axis of which lies on the xz plane and is parallel with the X_{03} axis.

It may be noted that the motion comes to rest in the phase space at the points where the rotary elliptic paraboloid is tangent with the sphere of Eq. (19). These points, referred to as critical points, can be obtained by equating the right-hand side of Eq. (11) to zero, the number of which differs from 2 to 4, depending on the shape of the paraboloid.

In the case where three critical points exist as shown in Fig. 3, the hyperbolic cylinder

$$(d/2)x - axz - gz = 0 \tag{20}$$

which is obtained by setting the right-hand side of the equation of the second component in Eq. (11) to zero, is tangent with the sphere. In order for the plane [tangent with the sphere at the point $(x_1, 0, z_1)$ on the sphere]

$$x_1 x + z_1 z = 1 \qquad (0 \leqq x_1, z_1 \leqq 1) \tag{21}$$

to be tangent with the hyperbolic surface of Eq. (20), the following relation must hold, namely

$$\{a - gx_1 - (d/2)z_1\}^2 + 4agx_1 = 0 \tag{22}$$

This is derived from the quadratic equation obtained by the substitution of Eq. (21) into Eq. (20). The condition that the x coordinate $x = \{a - gx_1 - (d/2)z_1\}/2ax_1$ of the tangent point is equal to x_1 gives

$$x_1 = (-g/a)^{1/3} \tag{23}$$

with the aid of Eq. (22). Upon substituting Eq. (23) into Eq. (22) and rearranging, one obtains the condition for the number of critical points to be three as

$$(d/2)^{2/3} + (-g)^{2/3} = a^{2/3} \tag{24}$$

Inspection of the preceding discussion readily reveals that the conditions for two and four critical points to exist are, respectively,

$$(d/2)^{2/3} + (-g)^{2/3} \gtrless a^{2/3} \tag{25}$$

For the abovementioned case where $v_i = 0$ and $v_f = 2\pi$, the relation between the time t and the variable z is derived from the equation of the third component in Eq. (11), Eqs. (17) and (19);

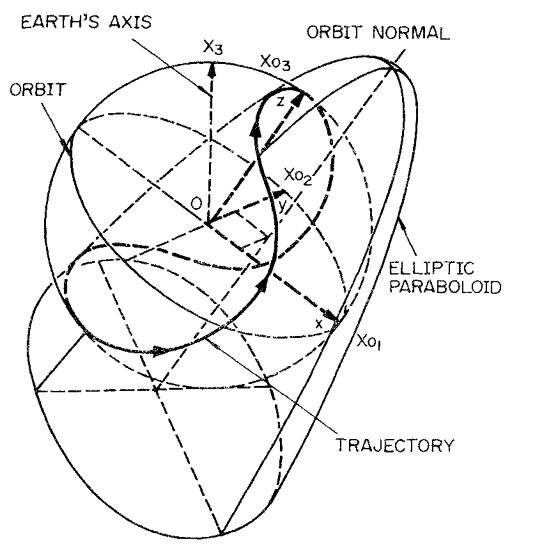

Fig. 2 Phase trajectory (for magnetic and gravitational torques).

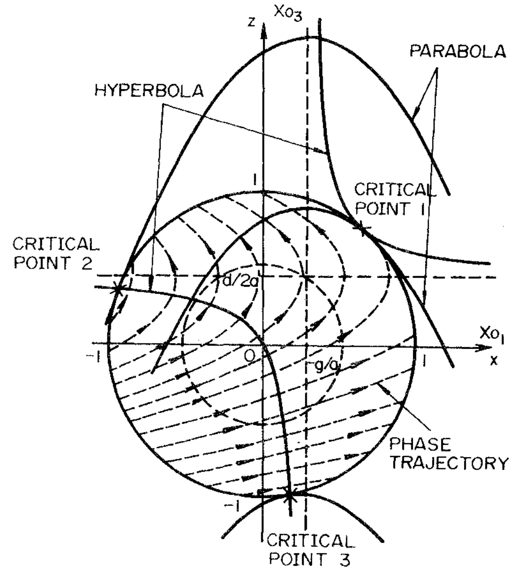

Fig. 3 The case where three critical points exist.

$$t = \pm \int \left[-\frac{(az^2 - dz - a - c)^2}{4} - g^2 z^2 + g^2 \right]^{1/2} dz \tag{26}$$

which represents an elliptic integral.[5]

3. Perturbation of Periodic Solution

In performing magnetic torquing, the magnetic torque is generally made dominant over the gravitational torque, as well as the other external torques. In a case where the gravitational torque is negligible compared to the magnetic torque, one obtains $a = b = h = 0$ by setting $G = 0$ in Eq. (14), and $\mathbf{g(x)} = 0$ from Eq. (13). In this case Eq. (11) is reduced to a linear system, which renders the general solution[6]

$$\mathbf{x} = (c_1 \cos a_m t - c_2 \sin a_m t)\mathbf{h}_1 + (c_1 \sin a_m t + c_2 \cos a_m t)\mathbf{h}_2 + c_3 \mathbf{h}_3 \tag{27}$$

in which

$$a_m = (g^2 + f^2 + d^2/4)^{1/2} \tag{28}$$

where c_1, c_2, and c_3 are integration constants, and $\mathbf{h}_1$, $\mathbf{h}_2$, and $\mathbf{h}_3$ are the characteristic vectors of the matrix A given by

$$\begin{bmatrix} \mathbf{h}_1 \\ \mathbf{h}_2 \\ \mathbf{h}_3 \end{bmatrix} = P \begin{bmatrix} \mathbf{X}_{01} \\ \mathbf{X}_{02} \\ \mathbf{X}_{03} \end{bmatrix} \tag{29}$$

where

$$P = \begin{bmatrix} gd/2a_m a_m', & -fd/2a_m a_m', & a_m'/a_m \\ f/a_m', & -g/a_m', & 0 \\ g/a_m, & f/a_m, & d/2a_m \end{bmatrix} \tag{30}$$

and

$$a_m' = (g^2 + f^2)^{1/2} \tag{31}$$

This result shows that the spin vector rotates about the vector $\mathbf{h}_3$ at the constant speed a_m, so that the phase trajectory is formed by the intersection of the sphere and the plane as shown in Fig. 4.

We now analyze the perturbation effects of the gravitational torque on the periodic motion due to the dominant magnetic torque. Upon performing the transformation defined by

$$\mathbf{X} = P\mathbf{x} = X\mathbf{h}_1 + Y\mathbf{h}_2 + Z\mathbf{h}_3 \tag{32}$$

$$u = a_m t \tag{33}$$

Eq. (11) becomes

$$\dot{\mathbf{X}} = d\mathbf{X}/du = B\mathbf{X} + \mu\mathbf{f}(\mathbf{X}) \tag{34}$$

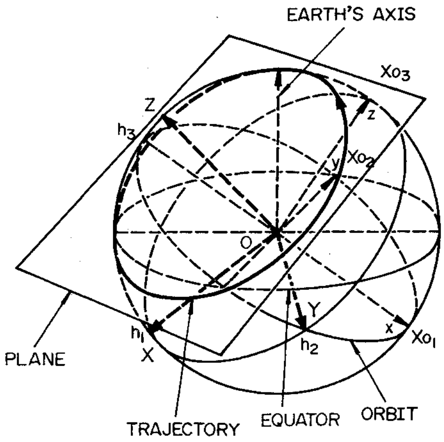

Fig. 4 Phase trajectory (for magnetic torque).

where the matrix B is in canonical form

$$B = \frac{PAP^{-1}}{a_m} = \begin{bmatrix} 0, & -1, & 0 \\ 1, & 0, & 0 \\ 0, & 0, & 0 \end{bmatrix} \tag{35}$$

and

$$\mathbf{f}(\mathbf{X}) = \begin{bmatrix} f_1(\mathbf{X}) \\ f_2(\mathbf{X}) \\ f_3(\mathbf{X}) \end{bmatrix} = \begin{bmatrix} (a_m' KY + a_m LZ)/a_m{}^2 a_m' \\ K(a_m' X + (d/2)Z)/a_m{}^2 a_m' \\ (a_m LX - (d/2)KY)/a_m{}^2 a_m' \end{bmatrix} \tag{36}$$

$$\begin{cases} K = (dF/2a_m a_m')X - (J/a_m')Y - (F/a_m)Z \\ L = (dJ/2a_m a_m')X - (H/a_m')Y - (J/a_m)Z \end{cases} \tag{37}$$

and

$$\begin{cases} F = bf^2 + 2fgh + ag^2 \\ J = (f^2 - g^2)h - (a-b)fg \\ H = af^2 - 2fgh + bg^2 \end{cases} \tag{38}$$

The constant μ in Eq. (34) is employed for the purpose of representing a small quantity.

The perturbed motion will be analyzed by employing Poincaré's method concerning approximation of periodic solutions for nonlinear systems.[7] The results obtained in the previous chapter show that Eq. (34) has a periodic solution whose phase trajectory is formed by the intersection of the elliptic paraboloid of Eq. (17) with the sphere of Eq. (19). Hence, noticing that Eq. (34) with $\mu = 0$ has a solution $e^{uB}\mathbf{c}$ where $\mathbf{c} = c_1\mathbf{h}_1 + c_2\mathbf{h}_2 + c_3\mathbf{h}_3$, one may suppose that Eq. (34) with small μ also has a periodic solution with the same initial value $\mathbf{c}$. Let this solution be $\boldsymbol{\phi} = \boldsymbol{\phi}(u, \mu, \mathbf{c})$, where $\boldsymbol{\phi}(0, \mu, \mathbf{c}) = \mathbf{c}$, which is continuous in μ for μ near $\mu = 0$. Then from Eq. (34), using the variation-of-constants formula, one obtains

$$\boldsymbol{\phi}(u, \mu, \mathbf{c}) = e^{uB}\mathbf{c} + \mu \int_0^u e^{(u-s)B}\mathbf{f}(\boldsymbol{\phi}(s, \mu, \mathbf{c}))\, ds \tag{39}$$

The necessary and sufficient condition for $\boldsymbol{\phi}$ to be periodic of period $(2\pi + \tau)$ is

$$\boldsymbol{\phi}(0, \mu, \mathbf{c}) = \boldsymbol{\phi}(2\pi + \tau, \mu, \mathbf{c}) \tag{40}$$

or

$$(e^{2\pi B} - E)\mathbf{c} + e^{2\pi B}(e^{\tau B} - E)\mathbf{c} + \mu \int_0^{2\pi + \tau} e^{(2\pi + \tau - s)B}\mathbf{f}(\boldsymbol{\phi}(s, \mu, \mathbf{c}))\, ds = 0 \tag{41}$$

The first term of the left-hand side of this equation is obviously zero.

The second component of $e^{uB}\mathbf{c}$ is $c_1 \sin u + c_2 \cos u$, and hence, for any specific choice of c_1 and c_2, this sinusoid vanishes for some value of u and has there a nonzero first derivative. By continuity, the second component ϕ_2 of $\boldsymbol{\phi}(u, \mu, \mathbf{c})$ must cross the u axis at some u also. The system (34) is invariant under translations in u because the system is autonomous. Hence, in

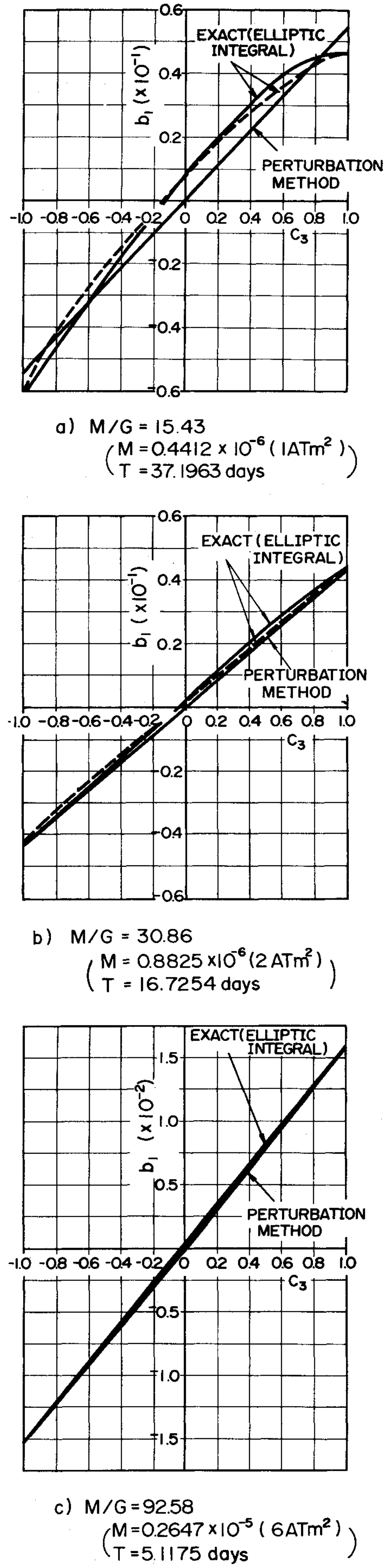

Fig. 5 Perturbed change in period due to small gravitational torque. $G = 0.2858 \times 10^{-7}$; ——— $c_1 > 0$, - - - - - - $c_1 < 0$, for exact values.

Table 1 Physical and orbital parameters of a model satellite

Moment of inertia	
about the spin axis I_z	0.673 kg m sec^2
trans. to the spin axis I_x	0.542 kg m sec^2
Spin velocity ω_s	10 rpm
Semimajor axis a	7496 km
Eccentricity e	0.11673
Inclination i	31.15°
Regression of the nodes	4.9735°/day

what follows it will be assumed that the second component ϕ_2 vanishes at $u = 0$. That is

$$\phi_2(0, \mu, \mathbf{c}) = c_2 = 0 \tag{42}$$

If $\tau = \tau(\mu)$ is continuous for small μ and $\tau(0) = 0$, then Eq. (41) divided by μ gives the following, when letting $\mu \to 0$:

$$\left[e^{2\pi B} B\mathbf{c} \lim_{\mu\to 0}\left(\frac{\tau}{\mu}\right) + \int_0^{2\pi} e^{(2\pi - s)B}\mathbf{f}(e^{sB}\mathbf{c})\,ds\right]_j = 0 \quad (j = 1, 2, 3) \tag{43}$$

where $[\ \ ]_j$ represents the jth component. Setting

$$b_1 = \frac{1}{2\pi}\lim_{\mu\to 0}\left(\frac{\tau}{\mu}\right) = \frac{v_0}{2\pi} \tag{44}$$

which represents the proportion of change in period due to the gravitational torque, one obtains from Eq. (43) for $j = 2$:

$$b_1 = -\frac{1}{2\pi c_1}\int_0^{2\pi}\{-f_1(e^{sB}\mathbf{c})\sin s + f_2(e^{sB}\mathbf{c})\cos s\}\,ds =$$
$$-c_3[3\{(g^2-f^2)a - 2fgh\} - (a_m{}^2 - 3f^2)(a+b)]/2a_m{}^3 \tag{45}$$

This result shows that the perturbation in period is proportional to the projection of the spin vector on the characteristic vector $\mathbf{h}_3$, about which the spin vector rotates.

Figure 5 shows, as an example for the case where the external torques are averaged over an entire orbit, computed values b_1 for a model satellite whose orbital and physical parameters are given in Table 1. The figure shows, in addition to the approximate values computed from the previously derived Eq. (45), the exact ones obtained from Eqs. (26) and (27). Inspection of the figure indicates, as is expected, that the agreement of both values increases with increasing values of M/G.

We now consider the problem of expanding the periodic solution of period $(2\pi + \tau)$ obtained above, and τ in the power-series of μ. Designating the periodic solution as $\mathbf{q}(u, \mu)$, we replace u by s where

$$u = s(1 + \tau/2\pi) \tag{46}$$

and let

$$\mathbf{q}(s(1 + \tau/2\pi), \mu) = \mathbf{p}(s, \mu) \tag{47}$$

Clearly $\mathbf{p}$ is periodic in s of period 2π. As the function $\mathbf{f(X)}$ in Eq. (34) is analytic, the coefficients in the power-series expansions for $\mathbf{p}$ and τ

$$\mathbf{p}(s, \mu) = \sum_{i=0}^{\infty}\mu^i\mathbf{p}^{(i)}(s) = \mathbf{p}^{(0)}(s) + \mu\mathbf{p}^{(1)}(s) + \mu^2\mathbf{p}^{(2)}(s) + \cdots \tag{48}$$

$$\tau/2\pi = \sum_{i=1}^{\infty}\mu^i b_i = \mu b_1 + \mu^2 b_2 + \cdots \tag{49}$$

can be calculated recursively where both series converge for small μ. Clearly, $2\pi b_1 = v_0$, and $\mathbf{p}^{(0)}(s) = e^{sB}\mathbf{c}$. Since the second component of $\mathbf{q}(0, \mu)$, namely c_2, is equal to zero, it follows that this component of $\mathbf{p}^{(i)}(0)$ must also be zero for all $i \geqq 0$.

Upon substituting Eqs. (48) and (49) into Eq. (34), and by comparing coefficients of the powers of μ, the following system of equations is obtained:

$$\left\{\begin{aligned} &d\mathbf{p}^{(0)}/ds = B\mathbf{p}^{(0)} \\ &d\mathbf{p}^{(1)}/ds = B\mathbf{p}^{(1)} + b_1 B\mathbf{p}^{(0)} + \mathbf{f}(\mathbf{p}^{(0)}) \\ &d\mathbf{p}^{(2)}/ds = B\mathbf{p}^{(2)} + b_1 B\mathbf{p}^{(1)} + b_2 B\mathbf{p}^{(0)} + b_1\mathbf{f}(\mathbf{p}^{(0)}) + \mathbf{f}_x(\mathbf{p}^{(0)})\mathbf{p}^{(1)} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &d\mathbf{p}^{(j)}/ds = B\mathbf{p}^{(j)} + b_1 B\mathbf{p}^{(j-1)} + b_j B\mathbf{p}^{(0)} + \mathbf{f}_x(\mathbf{p}^{(0)})\mathbf{p}^{(j-1)} + \mathbf{F}^{(j)} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}\right. \tag{50}$$

where $\mathbf{F}^{(j)}$ depends on $\mathbf{p}^{(0)}, \mathbf{p}^{(1)}, \ldots, \mathbf{p}^{(j-2)}$, and $b_1, b_2, \ldots, b_{j-1}$. The $\mathbf{p}^{(i)}$ and b_i are uniquely determined in succession in Eq. (50) by the requirements that $2\pi b_1 = v_0$, $\mathbf{p}^{(0)}(s) = e^{sB}\mathbf{c}$, $p_2{}^{(i)}(0) = 0$, and $\mathbf{p}^{(i)}(s + 2\pi) = \mathbf{p}^{(i)}(s)$.

Here we determine $\mathbf{p}^{(1)}$. Using the variation-of-constants formula, one has

$$\mathbf{p}^{(1)}(s) = e^{sB}\mathbf{c}^{(1)} + \int_0^s e^{(s-t)B}[b_1 B\mathbf{p}^{(0)} + \mathbf{f}(\mathbf{p}^{(0)})]\,dt \tag{51}$$

where $\mathbf{c}^{(1)}$ vanishes because $\mathbf{p}(s, \mu)$ is identical with $\mathbf{p}^{(0)}(s)$ at $s = 0$.

The evaluation of the above integral can be carried through without difficulty, with the aid of Eqs. (36–38). The detailed steps will not be reproduced here. Only the end result for the third component will be given, for want of space:

$$\begin{aligned} p_3{}^{(1)}(s) = {} & (c_1/a_m{}^2 a_m{}'^2)\,|\,(f^2 - g^2)dhc_1 s/2 + (a-b)dfgc_1\sin 2s/4 - \\ & [(a_m{}^2 + d^2/4)\{(f^2-g^2)a - 2fgh\} + (a_m{}^2 g^2 - d^2 f^2/4)(a+b)]\cdot \\ & c_1\sin^2 s - a_m{}'\{(f^2-g^2)h + (a-b)fg\}c_3\sin s - (a_m{}'d/2a_m)\cdot \\ & \{(f^2-g^2)a - 2fgh - (a+b)f^2\}c_3(\cos s - 1)\,| \end{aligned} \tag{52}$$

In the case where $v_i = 0$ and $v_f = 2\pi$, this becomes

$$p_3{}^{(1)}(s) = (\pi a_m{}'^2 c_1/2a_m{}^3)[-c_1\sin^2 s + (dc_3/a_m{}')(\cos s - 1)] \tag{53}$$

Table 2 shows computed values of $p_3{}^{(1)}(s)$ for the model

Table 2 Perturbed change in trajectory due to small gravitational torque: $G = 0.2858 \times 10^{-7}$

a) $M/G = 15.43[M = 0.4412 \times 10^{-6}$ (1 atm^2), $c_1 > 0]$

s	0°	30°	60°	90°	120°	150°	180°
$p_3{}^{(1)}(s)$	0	−0.00693	−0.02079	−0.02773	−0.02079	−0.00693	0
Exact solution	0	−0.00654	−0.02010	−0.02769	−0.02150	−0.00736	0
$p_3{}^{(1)}(s)$	0.5	0.49872	0.49903	0.50846	0.52829	0.54940	0.55851
Exact solution	0.5	0.49874	0.49902	0.50883	0.53047	0.55453	0.56519
$p_3{}^{(1)}(s)$	0.93969	0.94179	0.94812	0.95816	0.96983	0.97940	0.98312
Exact solution	0.93969	0.94192	0.94871	0.95973	0.97283	0.98378	0.98808

b) $M/G = 92.58[M = 0.2647 \times 10^{-5}$ (6 atm^2), $c_1 > 0]$

s	0°	30°	60°	90°	120°	150°	180°
$p_3{}^{(1)}(s)$	0	−0.00027	−0.00082	−0.00110	−0.00082	−0.00027	0
Exact solution	0	−0.00027	−0.00082	−0.00110	−0.00083	−0.00027	0
$p_3{}^{(1)}(s)$	0.5	0.50015	0.50070	0.50180	0.50332	0.50470	0.50525
Exact solution	0.5	0.50015	0.50070	0.50182	0.50336	0.50476	0.50533
$p_3{}^{(1)}(s)$	0.93969	0.93992	0.94057	0.94151	0.94252	0.94330	0.94359
Exact solution	0.93969	0.93993	0.94059	0.94155	0.94258	0.94337	0.94367

satellite shown in Table 1. In Table 2 the exact values are also shown. They are computed from Eq. (26) and the equation

$$a(1-z^2)+2gx+dz+c=0 \tag{54}$$

which is derived from Eqs. (17) and (19).

4. Conclusions

The effects of the magnetic and gravitational torques on the rotational motion of a spinning rigid body have been investigated. An uncanted dipole model was used to represent the Earth's magnetic field, and the angular momentum vector was assumed to be aligned with the spin axis at all times. As a result of these assumptions, the attitude of a uniaxial satellite responding to these torques averaged over specified portions of an eccentric orbit can be described by an autonomous system of nonlinear differential equations relative to the orbit fixed coordinate system.

In order to show that the motion is periodic, the equations of motion have first been integrated after some manipulations to yield the phase trajectory, which is formed by the intersection of an elliptic paraboloid and a sphere. Then the conditions to determine the number of equilibrium points where the motion comes to rest have been derived for the case where the external torques are averaged over an entire orbit.

Finally, the perturbation effects of the gravitational torque on the attitude variation under the dominant influence of the magnetic torque have been investigated, by applying Poincaré's method concerning approximation of periodic solutions for non-linear systems. It is found that the perturbed change in period is proportional to the projection of the spin vector on the characteristic vector, about which the spin vector rotates, of the linear system to which the system reduces when only the magnetic torque exists. The approximate analytic solution of the first order for the perturbed motion has also been obtained as the coefficient of the power-series expansion in a small parameter μ, representing the perturbing effect of the gravitational torque. These results clarify in a more general form the influence of physical and orbital parameters of the system upon the gyroscopic behavior of a spinning satellite.

References

[1] Beletsky, V. V., "Motion of an Artificial Earth Satellite About Its Center of Mass," *Artificial Earth Satellites*, Vols. 1–2, Plenum, New York, 1960, pp. 30–54.

[2] Beletsky, V. V., *Motion of an Artificial Satellite About Its Center of Mass*, Israel Program for Scientific Translations, Jerusalem, 1966.

[3] Beletsky, V. V., "New Investigations of Motion of an Artificial Satellite About Its Center of Mass," *Dynamics of Satellites*, Springer-Verlag, Berlin, 1970, pp. 194–206.

[4] Colombo, G., "On the Motion of Explorer XI Around Its Center of Mass," *Torques and Attitude Sensing in Earth Satellites*, Academic Press, New York, 1964, pp. 175–190.

[5] Holland, R. L., "A First-Order Theory for the Rotational Motion of a Triaxial Rigid Body Orbiting an Oblate Primary," *Astronomical Journal*, Vol. 74, No. 3, April 1969, pp. 490–496.

[6] Hara, M. and Sawano, S., "Analysis of the Rotational Motion of a Uniaxial Satellite Due to Magnetic and Gravitational Torques," *Astronautical Research* 1970, *Proceedings of the 21st IAC*, North-Holland, the Netherlands, 1971, pp. 59–77.

[7] Coddington, E. A. and Levinson, N., *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955, pp. 348–370.